

# BALLISTIC DYNAMICS OF DIRAC PARTICLES IN ELECTRO-MAGNETIC FIELDS

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**ABSTRACT.** Investigating properties of two-dimensional Dirac operators coupled to an electric and a magnetic field (perpendicular to the plane) requires in general unbounded (vector-) potentials. If the system has a certain symmetry, the fields can be described by one-dimensional potentials  $V$  and  $A$ . Assuming that  $|A| < |V|$  outside some arbitrary large ball, we show that absolutely continuous states of the effective Dirac operators spread ballistically. These results are based on well-known methods in spectral dynamics together with certain new Hilbert-Schmidt bounds. We use Lorentz boosts to derive these new estimates.

## 1. INTRODUCTION

It is well known that Dirac particles suffer from a phenomenon called Klein tunneling. In dimension one, it can be roughly described as follows : If one considers a step potential, for instance  $V(x) = V_0$  for  $x \geq 0$  and zero otherwise, then massless Dirac particles coming from the left will tunnel through the barrier independently of their energy. As opposed to the classical quantum tunneling there is no exponential damping factor diminishing the probability of finding the particle on the right side of the barrier [9, 20]. More generally, one-dimensional massless Dirac particles spread as free particles in the presence of electric fields. This effect has attracted renewed attention due to the isolation of graphene in 2003 (see [12]), since the low-energy charge carriers of this material can be described by the two-dimensional massless Dirac equation [1, 4, 5]. Indeed, experiments have been carried out to observe Klein tunneling in graphene confirming some theoretical predictions [8, 19, 24].

Consider the massless one-dimensional Dirac equation

$$-i\sigma_1\partial_1 + V \quad \text{on} \quad L^2(\mathbb{R}, \mathbb{C}^2),$$

with an electric potential  $V \in L^1_{\text{loc}}(\mathbb{R})$ , where  $\sigma_1$  is the first Pauli matrix. In this case the Klein tunnel effect is not very surprising from the mathematical point of view since  $-i\sigma_1\partial_1 + V$  is unitarily equivalent to the free Dirac operator  $-i\sigma_1\partial_1$  by means of the transformation

$$(1) \quad \exp\left(i\sigma_1 \int_0^x V(s)ds\right).$$

However, in the presence of magnetic fields the situation is different. In dimension two it is known that magnetic fields tend to localise Dirac particles, very much like as in the Schrödinger case (see [20]).

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In a previous article we considered the combined electromagnetic effect from a spectral theoretical point of view [11]. In the present work we investigate this further but focusing on the wave package spreading. Consider a two-dimensional Dirac operator coupled to an electro-magnetic field described by electric and magnetic potentials  $V$  and  $\mathbf{A}$ . If the field has translational or rotational symmetry the problem can be reduced to the study of a family of Dirac operators on the line or on the half-line, respectively. (Here the fields may be expressed through one-dimensional potential functions  $V$  and  $A$ .) Denote by  $h$  one of the members of these families. Our results roughly state the following: Assuming that the function  $\psi \neq 0$  is of finite energy and that it belongs to the absolutely continuous spectral subspace of  $h$  we obtain a lower bound on the Cèsaro mean of the time evolution of the  $p$ -th moment ( $p > 0$ ), i.e. there is a constant  $C(\psi, p) > 0$  such that

$$(2) \quad \frac{1}{T} \int_0^T \| |x|^{p/2} e^{-i t h} \psi \|^2 dt \geq C(\psi, p) T^p.$$

Besides certain regularity conditions the above inequality holds provided  $|A| < |V|$  outside some arbitrary large ball (see Theorems 2 and 4). As a consequence of the causal behaviour of Dirac particles (see [20, Theorem 8.5]) one has an upper bound of the same type, yielding altogether ballistic dynamics. The consequences of inequalities of type (2) for two-dimensional Dirac operators with symmetries are summarised in Corollaries 1 and 2. We remark that if  $V$  grows regularly at infinity the spectrum of  $h$  is purely absolutely continuous (see the discussion after Remark 2). The latter is in stark contrast to the behaviour of non-relativistic particles. An important example is when the electric and magnetic fields are asymptotically uniform, in which case  $V$  and  $A$  grow linearly in the space coordinate.

The proof of the bounds of type (2) are based upon the ideas of [7], [2] and [10]. These results say roughly the following: Let  $K \subset \mathbb{R}$  be a compact set and  $\mathbb{1}_K$  be the characteristic function supported in  $K$ . Then, the inequality (2) holds if the function  $\psi \in \mathbb{1}_K(h)L^2$  belongs to the absolutely continuous subspace of  $h$  provided a certain Hilbert-Schmidt bound is verified. This latter condition demands the following for the product of characteristic functions in space and energy: There is a constant  $C > 0$  such that for all  $I \subset \mathbb{R}$  compact

$$(3) \quad \|\mathbb{1}_I(x)\mathbb{1}_K(h)\|_{\text{HS}} \leq C_K \sqrt{|I|}.$$

It is easy to check that the required bound is satisfied for the free Dirac operator. For Schrödinger operator with potentials bounds like (3) are obtained using semigroup properties combined with perturbation theory [18]. However, in our case there is no proper semi-group theory and, in addition, when the potentials are allowed to grow at infinity, naive (resolvent) perturbation theory gives estimates where the scaling in  $|I|$  depends on the growth rate of  $A$  and  $V$ ; that would eventually not deliver (2). This is not surprising since in this case  $A$  and  $V$  should not be treated as perturbations to the free Hamiltonian. In the case  $A = 0$  one easily sees that the transformation (1) solves this problem. In this work we provide new estimates of the type (3) for the general case  $V, A \neq 0$  as long as  $A$  is dominated by  $V$  in certain sense. Our approach is to use Lorentz boosts (of non-constant speed) to transform the Hamiltonian to another operator with a magnetic vector potential that vanishes at infinity. We remark that the transformed operator is not going to be symmetric (see Section 4) since Lorentz boosts are not represented through unitary maps in  $L^2$  but only through invertible transformation (see [20, p. 70]). The

relation between the original Hamiltonian and the Lorentz transformed operator is made precise through certain resolvent identities. In the case of operators defined on the real line bounds like (2) are very much a corollary of (3) and the proof of [10, Theorem 6.2]. However, for Dirac operator defined on the half-line one should proceed more carefully due to their singularities at zero (c.f., Remark 1 and the discussion at the beginning of Section 6).

*This article is organised as follows:* In the next section we state precisely our main results. The definition and basic properties of the one-dimensional Dirac operators used here can be found in Section 3. In Section 4 we discuss the behaviour of Dirac operators under certain Lorentz boosts of non-constant speed and establish resolvent identities between original and transformed operators. We then apply the insight of Section 4 to prove Theorems 1 and 3 in Section 5. The dynamical bounds for the half-line operators (Theorem 4) are proven in Section 6 where we also establish a local compactness property suitable for our regularity assumptions. In Appendix A we collect some technical facts concerning self-adjointness. We compute the resolvent kernel for a half-line operator in Appendix B. Finally, in Appendix C we prove Corollaries 1 and 2 about the consequences for two-dimensional Dirac operator with symmetries.

## 2. MAIN RESULTS AND ITS APPLICATIONS

The massless two-dimensional Dirac operator in an electromagnetic field described by an electric potential  $V$  and a magnetic field  $B$  (perpendicular to the plane) is given by

$$(4) \quad H = \boldsymbol{\sigma} \cdot (-i \nabla - \mathbf{A}) + V \quad \text{on} \quad \mathcal{H} := L^2(\mathbb{R}^2, \mathbb{C}^2).$$

Here  $\mathbf{A} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a magnetic vector potential satisfying  $B = \text{curl } \mathbf{A} := \partial_1 A_2 - \partial_2 A_1$  and  $\boldsymbol{\sigma} = (\sigma_1, \sigma_2)$  denote the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

For a rotational symmetric magnetic field  $B$  one can always choose the rotational gauge

$$(5) \quad \mathbf{A}(\mathbf{x}) = \frac{1}{r^2} \int_0^r B(s) s \, ds \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} =: \frac{A(r)}{r} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix},$$

where  $r = |\mathbf{x}|$ . Thus, if  $B$  and  $V$  are rotationally symmetric we can decompose  $H$  as a direct sum operators defined on the half-line, i.e.

$$(6) \quad H \cong \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} h_k,$$

where

$$(7) \quad h_k := -i \sigma_1 \partial_r + \sigma_2 \left( \frac{k}{r} - A(r) \right) + V(r) \quad \text{on} \quad L^2(\mathbb{R}^+, \mathbb{C}^2).$$

Here in a slight abuse of notation we write  $V(|\mathbf{x}|) = V(\mathbf{x})$ .

On the other hand, if  $B$  is translational symmetric, say, in the  $x_2$ -direction, we can choose the Landau gauge  $\mathbf{A}(\mathbf{x}) = (0, A(x_1))$ , where

$$(8) \quad A(x_1) = \int_0^{x_1} B(s) ds.$$

Thus, if in addition  $V$  has the same symmetry, the Hamiltonian  $H$  can be represented as a direct integral of one-dimensional fiber hamiltonians

$$(9) \quad H \cong \int_{\mathbb{R}}^{\oplus} h(\xi) d\xi,$$

with

$$(10) \quad h(\xi) = -i\sigma_1\partial_1 + \sigma_2(\xi - A) + V \quad \text{on} \quad L^2(\mathbb{R}, \mathbb{C}^2).$$

One-dimensional Dirac operators of type (7) and (10) are the object of the next four theorems. Let us fix some notation: Throughout this article we denote by  $\mathbb{1}_K$  the characteristic function on a set  $K \subset \mathbb{R}$ . We write  $P_{ac}(H)$  for the projection onto the absolutely continuous subspace associated to a self-adjoint operator  $H$ . We will make use of standard notation for norms:  $\|\cdot\|_p$  denotes the  $L^p$ -norm,  $\|\cdot\|_T$  is the graph norm with respect to an operator  $T$ , and  $\|\cdot\|_{\text{HS}}$  stands for the Hilbert-Schmidt norm.

In order to perform the afore mentioned Lorentz boosts (essentially of velocity  $A/V$ ) we introduce the following classes of electromagnetic potentials:

**Hypothesis 1 (H1).**  $A, V \in L^p_{\text{loc}}(\mathbb{R}, \mathbb{R})$  with  $p \geq 2$  such that  $A = A_1 + A_2$ ,  $V = V_1 + V_2$ , where  $A_1, V_1$  have compact support and  $A_2, V_2 \in C^1(\mathbb{R}, \mathbb{R})$  fulfill

- i)  $V_2$  is supported away from 0 and  $\text{supp}(A_2) \subset \text{supp}(V_2)$ ,
- ii)  $\|A_2/V_2\|_{\infty} < 1$ ,
- iii) the derivative  $(A_2/V_2)'$  is bounded on  $\mathbb{R}$ .

**Hypothesis 2 (H2).**  $A, V \in L^p_{\text{loc}}([0, \infty), \mathbb{R})$  with  $p > 2$  such that  $A = A_1 + A_2$ ,  $V = V_1 + V_2$ , where  $A_1, V_1$  have compact support and  $A_2, V_2 \in C^1(\mathbb{R}^+, \mathbb{R})$  fulfill

- i)  $V_2$  are supported away from 0 and  $\text{supp}(A_2) \subset \text{supp}(V_2)$ ,
- ii)  $\|A_2/V_2\|_{\infty} < 1$ ,
- iii) the derivative  $(A_2/V_2)'$  is bounded on  $[0, \infty)$ .

**Theorem 1.** For  $\xi \in \mathbb{R}$  let  $h(\xi)$  be given as in (10) with  $A, V$  satisfying Hypothesis (H1). Then there is a constant  $C_{\xi} > 0$  such that for any compact interval  $I \subset \mathbb{R}$  we have

$$\|\mathbb{1}_I(h(\xi) - i)^{-1}\|_{\text{HS}} \leq C_{\xi} \sqrt{|I|}.$$

A direct consequence of this HS-bound is the following theorem whose proof is the same as the one of Theorem 6.2 of [10].

**Theorem 2.** Consider the operator  $h(\xi)$ ,  $\xi \in \mathbb{R}$ , with  $A, V$  satisfying Hypothesis (H1). Let  $\Delta \subset \mathbb{R}$  be a bounded energy interval and  $\psi \in P_{ac}(h(\xi))\mathbb{1}_{\Delta}(h(\xi))L^2(\mathbb{R}, \mathbb{C}^2)$  be non-zero. Then, for each  $p > 0$ , there is a constant  $C_{\xi}(\psi, \Delta, p)$  such that

$$(11) \quad \langle \|x^{p/2} e^{-iTh(\xi)} \psi\|^2 \rangle_T \geq C_{\xi}(\psi, \Delta, p) T^p$$

for all  $T > 0$ .

In the case of the half-line operators  $h_k$  we obtain similar results:

**Theorem 3.** For  $k \in \mathbb{Z} + \frac{1}{2}$  let  $h_k$  be given as in (7), with  $A, V$  satisfying Hypothesis (H2). Then there is a constant  $C_k > 0$  such that for any compact interval  $I \subset [1, \infty)$  we have

$$\|\mathbb{1}_I(h_k - i)^{-1}\|_{\text{HS}} \leq C_k \sqrt{|I|}.$$

**Remark 1.** Note that the operators  $h_k$  have a  $k/x$ -singularity at zero. We do not need boundary conditions to define them. To deduce the HS-bounds for  $h_k$  we compare them with an auxiliary operator, which is regular at zero, and satisfies certain boundary conditions. Because of that, we obtain the HS-bounds only for intervals  $I \subset \mathbb{R}$  supported away from zero.

A consequence of the latter result is the following:

**Theorem 4.** Consider the operator  $h_k$ ,  $k \in \mathbb{Z} + \frac{1}{2}$  with  $A, V$  satisfying Hypothesis (H2). Let  $\Delta \subset \mathbb{R}$  be a bounded energy interval and  $\psi \in P_{ac}(h_k)\mathbb{1}_\Delta(h_k)L^2(\mathbb{R}^+, \mathbb{C}^2)$  be non-zero. Then, for each  $p > 0$ , there is a constant  $C_k(\psi, \Delta, p)$  such that

$$(12) \quad \langle \|x^{p/2} e^{-iTh_k} \psi\|^2 \rangle_T \geq C_k(\psi, \Delta, p) T^p$$

for all  $T > 0$ .

This statement is proven in Section 6. We have to modify the argument of [10] since Theorem 3 is only valid for intervals with non-vanishing distance to zero. As an additional ingredient we use the local compactness of  $h_k$  (also proven in Section 6) for  $A, V \in L^p_{\text{loc}}([0, \infty), \mathbb{R})$  with  $p > 2$ .

**Remark 2.** We note that theorems 1 - 4 also hold for massive Dirac operators, i.e. operators of the form  $h(\xi) + m\sigma_3$  or  $h_k + m\sigma_3$  with a constant  $m$ . Here  $\sigma_3 = -i\sigma_1\sigma_2$  is the third Pauli matrix.

Since Theorems 2 and 4 apply only if we have some absolutely continuous spectrum we mention some interesting examples: Let  $A, V$  satisfy (H1) and, in addition,

- $(A_2/V_2)'$  is integrable at  $\pm\infty$ ,
- $|V_2(x)| \rightarrow \infty$  as  $|x| \rightarrow \infty$ ,

then  $h(\xi)$ ,  $\xi \in \mathbb{R}$ , has purely absolutely continuous spectrum with  $\sigma_{ac}(h(\xi)) = \mathbb{R}$ . Similarly, if the potentials  $A, V$  satisfy (H2) and

- $(A_2/V_2)'$  is integrable at  $\infty$ ,
- $|V_2(x)| \rightarrow \infty$  as  $x \rightarrow \infty$ ,

then  $h_k$ ,  $k \in \mathbb{Z} + \frac{1}{2}$ , has purely absolutely continuous spectrum with  $\sigma_{ac}(h_k) = \mathbb{R}$  (see [15], Propositions 1 and 2).

Finally we illustrate how one can harness Theorems 2 and 4 to Dirac operators in dimension two.

**Corollary 1.** Let  $H$  be given as in (4) with translational symmetric  $B$  and  $V$ . Let  $A(x) := \int_0^x B(s)ds$  and  $V$  satisfy (H1). Let  $\Delta \subset \mathbb{R}$  be bounded and  $p > 0$ . Then for any  $\psi \in \mathbb{1}_\Delta(H)\mathcal{H}$  such that the set

$$(13) \quad \{\xi \in \mathbb{R} \mid \widehat{\psi}(\cdot, \xi) \neq 0, \widehat{\psi}(\cdot, \xi) \in P_{ac}(h(\xi))L^2(\mathbb{R}, \mathbb{C}^2)\},$$

has non-trivial Lebesgue measure, there exist a constant  $C(\psi, \Delta, p) > 0$  such that

$$\langle \|x_1|^{p/2} e^{-iTH} \psi\|^2 \rangle_T \geq C(\psi, \Delta, p) T^p$$

for all  $T > 0$ . Here  $\widehat{\psi}$  denotes the Fourier-transform of  $\psi$  in the  $x_2$ -variable, i.e. we use the notation  $\widehat{\psi}(x_1, \cdot) = \mathcal{F}_{x_2}\psi(x_1, \cdot)$ .

**Corollary 2.** *Let  $H$  be given as in (4) with rotational symmetric  $B$  and  $V$ . Let  $A(x) := x^{-1} \int_0^x B(s) ds$  and  $V$  satisfy (H2). Let  $\Delta \subset \mathbb{R}$  be bounded and  $p > 0$ . Then for  $\psi \in P_{ac}(H)\mathcal{H} \cap \mathbb{1}_\Delta(H)\mathcal{H}$ , with  $\psi \neq 0$ , there exist a constant  $C(\psi, \Delta, p) > 0$  such that*

$$\langle ||\mathbf{x}|^{p/2} e^{-itH} \psi ||^2 \rangle_T \geq C(\psi, \Delta, p) T^p$$

for all  $T > 0$ .

### 3. BASIC PROPERTIES OF DIRAC OPERATORS IN DIMENSION ONE

In this article we basically work with two different types of one-dimensional Dirac operators. The first type is given by

$$(14) \quad h = \sigma_1(-i\partial_x) - \sigma_2 A + V \quad \text{on} \quad L^2(\mathbb{R}, \mathbb{C}^2),$$

with  $A, V \in L^2_{\text{loc}}(\mathbb{R}, \mathbb{R})$ . We note that  $h$  is in the limit point case at  $\pm\infty$  [23, Korollar 15.21]. Then according to [23, Chapter 15] (see also [22]) the operator is essentially self-adjoint on

$$(15) \quad \mathcal{D}_0(h) = \{\psi \in \mathcal{D}_{\max}(h) \mid \psi \text{ has compact support in } \mathbb{R}\},$$

where

$$(16) \quad \mathcal{D}_{\max}(h) = \{\psi \in L^2(\mathbb{R}, \mathbb{C}^2) \mid \psi \text{ abs. cont.}, h\psi \in L^2(\mathbb{R}, \mathbb{C}^2)\}.$$

In fact by Lemma 4 in Appendix A we know that  $h$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}, \mathbb{C}^2)$ . We denote its self-adjoint extension by  $h$  again.

The second type of operators is defined on the half-line

$$(17) \quad h_k = \sigma_1(-i\partial_x) + \sigma_2 \left(\frac{k}{x} - A\right) + V \quad \text{on} \quad L^2((0, \infty), \mathbb{C}^2),$$

with  $A, V \in L^2_{\text{loc}}([0, \infty), \mathbb{R})$  and  $k \in (\mathbb{Z} + \frac{1}{2}) \cup \{0\}$ . The maximal domain of  $h_k$  is given by

$$(18) \quad \mathcal{D}_{\max}(h_k) = \{\psi \in L^2(\mathbb{R}^+, \mathbb{C}^2) \mid \psi \text{ abs. cont.}, h_k \psi \in L^2(\mathbb{R}^+, \mathbb{C}^2)\}.$$

For these half-line operators we distinguish two cases: When  $|k| \geq \frac{1}{2}$  the operator  $h_k$  is in the limit point case at  $+\infty$  [23, Korollar 15.21] and in the limit point case at 0 (see Proposition 2 in Appendix A). Thus  $h_k$  is essentially self-adjoint on  $\mathcal{D}_0(h_k)$  (defined as in (15)). We denote its self-adjoint extension by  $h_k$  again. Using Lemma 4 in Appendix A it actually holds that  $C_0^\infty((0, \infty), \mathbb{C}^2)$  is also an operator core for  $h_k$ .

**Remark 3.** *We note that by [23, Satz 15.6] the domains of self-adjointness of  $h$  and  $h_k$ ,  $|k| \geq 1/2$ , coincide with their maximal domains.*

In the case when  $k = 0$  the operator is in the limit point case at  $+\infty$  and in the limit circle case at 0. According to the theory of Sturm-Liouville operators (see [23, Satz 15.12] or [22])  $h_0$  has a one-parameter family of self-adjoint realisations with corresponding domains

$$(19) \quad \mathcal{D}^\alpha(h_0) = \{\mathcal{D}_{\max}(h_0) \mid \lim_{x \rightarrow 0} \psi_1(x) \cos \alpha - \psi_2(x) \sin \alpha = 0\}, \quad \alpha \in [0, 2\pi).$$

In the sequel we work with the self-adjoint realisation of  $h_0$  on  $\mathcal{D}^0(h_0) \equiv \mathcal{D}(h_0)$  and denote, as before, the resulting operator by the same symbol.

**Remark 4.** Let  $\chi \in C^\infty((0, \infty), [0, 1])$  be a smooth function supported away from zero with bounded first derivative. By the definition of the domains of self-adjointness (18) (see Remark 3) and (19) and the fact that  $k/x$  is a bounded function on the support of  $\chi$  we have that

$$(20) \quad \chi \mathcal{D}(h_k) \subset \mathcal{D}(h_0),$$

whenever the two operators have the same potentials  $V$  and  $A$ .

**Proposition 1.** Consider  $h, h_0$  with  $V \in L^2_{\text{loc}}$  and  $A = 0$ . Then for bounded intervals  $I \subset \mathbb{R}$ ,  $I_0 \subset (0, \infty)$  the operators  $\chi_I(h - i)^{-1}$ ,  $\chi_{I_0}(h_0 - i)^{-1}$  are Hilbert-Schmidt with Hilbert-Schmidt norms

$$(21) \quad \left\| \chi_I \frac{1}{(h - i)} \right\|_{\text{HS}} \leq \frac{1}{\sqrt{2}} |I|^{1/2},$$

$$(22) \quad \left\| \chi_{I_0} \frac{1}{(h_0 - i)} \right\|_{\text{HS}} \leq |I_0|^{1/2}.$$

*Proof.* Intertwining the resolvents by the unitary transformation

$$(23) \quad [U\psi](x) = \exp\left(i\sigma_1 \int_0^x V(s)ds\right) \psi(x)$$

on  $L^2(\mathbb{R}, \mathbb{C}^2)$ , respectively on  $L^2((0, \infty), \mathbb{C}^2)$ , the proof reduces to the case  $V = 0$ . Then, the statement on  $\chi_I(h - i)^{-1}$  is a direct consequence of the Kato-Seiler-Simon inequality (see [16] or [17, Thm. 4.1]). For the claim on the operator  $\chi_{I_0}(h_0 - i)^{-1}$  on  $L^2((0, \infty), \mathbb{C}^2)$  we use directly the resolvent kernel (computed in the Appendix B) given by

$$(24) \quad \frac{1}{\sigma_1(-i\partial_x) - i}(x_1, x_2) = \begin{cases} ie^{-x_1} \begin{pmatrix} \sinh x_2 & \cosh x_2 \\ \sinh x_2 & \cosh x_2 \end{pmatrix} & \text{for } x_1 > x_2 \geq 0 \\ ie^{-x_2} \begin{pmatrix} \sinh x_1 & -\sinh x_1 \\ -\cosh x_1 & \cosh x_1 \end{pmatrix} & \text{for } x_2 > x_1 \geq 0. \end{cases}$$

Since  $\chi_{I_0}(x_1)(\sigma_1(-i\partial_x) - i)^{-1}(x_1, x_2)$  is square-integrable, we obtain that  $\chi_{I_0}(h_0 - i)^{-1}$  is a Hilbert-Schmidt operator. In fact this is true for any function  $f \in L^2((0, \infty))$ ; the norm may be computed as

$$(25) \quad \begin{aligned} \left\| f \frac{1}{(h_0 - i)} \right\|_{\text{HS}}^2 &= \int_0^\infty \int_0^\infty |f(x_1)|^2 \left\| \frac{1}{\sigma_1(-i\partial_x) - i}(x_1, x_2) \right\|_{M_2(\mathbb{C})}^2 dx_1 dx_2 \\ &= \int_0^\infty |f(x_1)|^2 \int_0^\infty e^{-2|x_1 - x_2|} dx_2 dx_1 \leq \|f\|_2^2. \end{aligned}$$

□

#### 4. LORENTZ TRANSFORMATIONS AND RESOLVENT IDENTITIES

It is known from the classical theory of electrodynamics that Lorentz boosts enables one to transform magnetic fields into electric ones and vice-versa. We are allowed to use this principle here, since the time dependent Dirac equation is invariant under Lorentz transformations (see Chapter 3 and Section 4.2 of [20]). In this section we use Lorentz boosts to transform the Hamiltonian to another operator whose magnetic vector potential vanishes at infinity. In our case the speed of the

boost is given by the ratio of  $A$  and  $V$  and will, therefore, depend on the space variable. Recall that in  $2 + 1$  space-time dimension a Lorentz boost in direction  $\mathbf{n} \in \mathbb{R}^2$  with speed  $\beta < 1$  (in general, smaller than the speed of light) is represented by the operator  $L_\Lambda = e^{\mathbf{n} \cdot \boldsymbol{\sigma} \theta / 2}$ , where  $\beta = \tanh \theta$ .

Let us point out a transformation property of the  $\sigma$ -matrices: Let  $a = 0$  or  $a = -\infty$  and  $\theta \in C^1((a, \infty), \mathbb{R})$ . Observe that

$$e^{-\sigma_2 \theta / 2} \sigma_1 e^{\sigma_2 \theta / 2} = e^{-\sigma_2 \theta} \sigma_1 = (\cosh \theta - \sigma_2 \sinh \theta) \sigma_1,$$

therefore,

$$\begin{aligned} e^{-\sigma_2 \theta / 2} \sigma_1 (-i \partial_1) e^{\sigma_2 \theta / 2} &= e^{-\sigma_2 \theta / 2} \sigma_1 e^{\sigma_2 \theta / 2} e^{-\sigma_2 \theta / 2} (-i \partial_1) e^{\sigma_2 \theta / 2} \\ (26) \quad &= (\cosh \theta - \sigma_2 \sinh \theta) \sigma_1 (-i \partial_1 - i \sigma_2 \frac{\theta'}{2}) \\ &= \cosh \theta (1 - \sigma_2 \tanh \theta) \sigma_1 (-i \partial_1 - i \sigma_2 \frac{\theta'}{2}) \end{aligned}$$

on the subspace  $C_0^\infty((a, \infty), \mathbb{C}^2)$ . Recall that for potentials  $V, A$  satisfying Hypothesis (H1) (for  $a = -\infty$ ) or (H2) (for  $a = 0$ ) the ratio fulfills  $\|A_2/V_2\|_\infty < 1$ . This enable us to define the following objects: Let  $\theta(x) = \tanh^{-1}(\beta(x))$ , where  $\beta := A_2/V = A_2/V_2$  and  $\gamma := \cosh \theta$  (then clearly,  $\gamma^{-1} = \sqrt{1 - \beta^2}$ ). For the Dirac operator  $h$  on  $L^2(\mathbb{R}, \mathbb{C}^2)$  (as given in (14)), with potentials satisfying (H1), we compute

$$\begin{aligned} e^{-\sigma_2 \theta / 2} h e^{\sigma_2 \theta / 2} &= \gamma (1 - \sigma_2 \beta) \sigma_1 (-i \partial_1 - i \sigma_2 \frac{\theta'}{2}) + (1 - \sigma_2 \frac{A_2}{V}) V - \sigma_2 A_1 \\ (27) \quad &= M \left[ \sigma_1 (-i \partial_1) + V/\gamma - \gamma (1 + \sigma_2 \beta) \sigma_2 A_1 + \sigma_3 \frac{\theta'}{2} \right] \end{aligned}$$

on  $C_0^\infty(\mathbb{R}, \mathbb{C}^2)$ , where  $M := \gamma(1 - \sigma_2 \beta)$  is a bounded multiplication operator with bounded inverse.

Analogously, for the operator  $h_k$  (defined in (17)), with potentials satisfying (H2), we have

$$(28) \quad e^{-\sigma_2 \theta / 2} h_k e^{\sigma_2 \theta / 2} = M \left[ \sigma_1 (-i \partial_1) + V/\gamma + \gamma (1 + \sigma_2 \beta) \sigma_2 \left( \frac{k}{x} - A_1 \right) + \sigma_3 \frac{\theta'}{2} \right]$$

on  $C_0^\infty((0, \infty), \mathbb{C}^2)$ . Note that, abusing notation, we use the symbol  $M$  in both cases, however, the former acts on  $L^2(\mathbb{R}^2, \mathbb{C}^2)$  and the latter on  $L^2((0, \infty), \mathbb{C}^2)$ . Summarising, we have the following identities on  $C_0^\infty(\mathbb{R}, \mathbb{C}^2)$  and  $C_0^\infty((0, \infty), \mathbb{C}^2)$ ,

$$(29) \quad e^{-\sigma_2 \theta / 2} h e^{\sigma_2 \theta / 2} = M \tilde{h} \quad \text{and} \quad e^{-\sigma_2 \theta / 2} h_k e^{\sigma_2 \theta / 2} = M \tilde{h}_k,$$

respectively. Where the operator

$$\tilde{h} := \sigma_1 (-i \partial_1) - \gamma (1 + \sigma_2 \beta) \sigma_2 A_1 + V/\gamma + \sigma_3 \frac{\theta'}{2} \quad \text{on } L^2(\mathbb{R}, \mathbb{C}^2),$$

is essentially self-adjoint on  $C_0^\infty(\mathbb{R}, \mathbb{C}^2)$ . Similarly,

$$\tilde{h}_k := \sigma_1 (-i \partial_1) + \gamma (1 + \sigma_2 \beta) \sigma_2 \left( \frac{k}{x} - A_1 \right) + V/\gamma + \sigma_3 \frac{\theta'}{2} \quad \text{on } L^2((0, \infty), \mathbb{C}^2),$$

is, for  $k \in \mathbb{Z} + \frac{1}{2}$ , in the limit point case at 0 and thus essentially self-adjoint on  $C_0^\infty((0, \infty), \mathbb{C}^2)$ . The latter holds since  $A_2 = 0$  in a vicinity of zero (and hence so is  $\beta$ ). Therefore,  $M \tilde{h}$  and  $M \tilde{h}_k$  are closed operators on the domains  $\mathcal{D}(\tilde{h})$  and  $\mathcal{D}(\tilde{h}_k)$ , respectively.

**Lemma 1.** *Consider the operator  $h$  and  $h_k$  with potentials satisfying Hypotheses (H1) and (H2), respectively. Then  $\mathcal{D}(\tilde{h}) = \mathcal{D}(M \tilde{h}) = e^{-\sigma_2 \theta / 2} \mathcal{D}(h)$ , respectively*



$\mathcal{D}(\tilde{h}_k) = \mathcal{D}(M\tilde{h}_k) = e^{-\sigma_2\theta/2}\mathcal{D}(h_k)$  for  $k \in \mathbb{Z} + \frac{1}{2}$ . In addition, the resolvent sets fulfill  $\varrho(h) = \varrho(M\tilde{h})$  and for any  $z \in \varrho(h)$  we have

$$(M\tilde{h} - z)^{-1} = e^{-\sigma_2\theta/2}(h - z)^{-1}e^{\sigma_2\theta/2},$$

$$\|(M\tilde{h} - z)^{-1}\| \leq \|(h - z)^{-1}\| \|e^{|\theta|}\|_\infty.$$

Similarly, for  $k \in \mathbb{Z} + \frac{1}{2}$ ,  $\varrho(h_k) = \varrho(M\tilde{h}_k)$  and for any  $z \in \varrho(h_k)$  holds

$$(M\tilde{h}_k - z)^{-1} = e^{-\sigma_2\theta/2}(h_k - z)^{-1}e^{\sigma_2\theta/2},$$

$$\|(M\tilde{h}_k - z)^{-1}\| \leq \|(h_k - z)^{-1}\| \|e^{|\theta|}\|_\infty.$$

*Proof.* We give the proof only for the operator  $h$ , since for  $h_k$  one can proceed analogously. The equality  $\mathcal{D}(\tilde{h}) = \mathcal{D}(M\tilde{h})$  is a direct consequence of the bounded invertibility of  $M$  (with inverse  $M^{-1} = \gamma(1 + \sigma_2\beta)$ ). Note that the relations (29) are also valid for  $C^1$ -functions with compact support, which form an invariant space under  $e^{\pm\theta/2\sigma_2}$  transformations. In addition, since  $C_0^1(\mathbb{R}, \mathbb{C}^2)$  is contained in  $\mathcal{D}(\tilde{h})$  and in  $\mathcal{D}(h)$ , it is also an operator core for  $\tilde{h}$  and  $h$ . Using (29) we easily get that, for any  $f \in C_0^1(\mathbb{R}, \mathbb{C}^2)$ ,

$$(30) \quad \|e^{-\sigma_2\theta/2}f\|_{M\tilde{h}} \leq \|e^{-\sigma_2\theta/2}\|_\infty \|f\|_h,$$

$$(31) \quad \|e^{\sigma_2\theta/2}f\|_h \leq \|e^{\sigma_2\theta/2}\|_\infty \|f\|_{M\tilde{h}}.$$

Let  $\varphi \in \mathcal{D}(h)$  and  $(\varphi_n)_{n \in \mathbb{N}} \subset C_0^1(\mathbb{R}, \mathbb{C}^2)$  be a sequence that converges to  $\varphi$  in the  $h$ -graph norm. Due to (30) the sequence  $(e^{-\sigma_2\theta/2}\varphi_n)_{n \in \mathbb{N}}$  is Cauchy in the  $M\tilde{h}$ -graph norm. Hence

$$\lim_{n \rightarrow \infty} e^{-\sigma_2\theta/2}\varphi_n = e^{-\sigma_2\theta/2}\varphi \in \mathcal{D}(M\tilde{h}).$$

Thus we get that  $e^{-\sigma_2\theta/2}\mathcal{D}(h) \subset \mathcal{D}(M\tilde{h})$ . The opposite inclusion can be shown along the same lines using the inequality (31).

In order to derive the resolvent bound observe that  $\mathcal{D}(M\tilde{h}) = e^{-\sigma_2\theta/2}\mathcal{D}(h)$  implies the operator identity, for any  $z \in \mathbb{C}$ ,

$$e^{-\sigma_2\theta/2}(h - z)e^{\sigma_2\theta/2} = (M\tilde{h} - z) \quad \text{on} \quad \mathcal{D}(M\tilde{h}).$$

Let  $z \in \varrho(h)$ . Since  $h - z : \mathcal{D}(h) \rightarrow L^2(\mathbb{R}, \mathbb{C}^2)$  is bijective with bounded inverse we conclude  $M\tilde{h} - z : \mathcal{D}(M\tilde{h}) \rightarrow L^2(\mathbb{R}, \mathbb{C}^2)$  is also bijective with bounded inverse. In addition,

$$\|(M\tilde{h} - z)^{-1}\| \leq \|e^{\theta/2\sigma_2}\| \|(h - z)^{-1}\| \|e^{-\theta/2\sigma_2}\|.$$

□

We close this section by illustrating how to deduce resolvent identities of the type presented in Lemma 1 when  $V < A$  at  $\infty$ . To this end we consider the operator  $h$  as given in (14) with  $A, V$  fulfilling

**Hypothesis 3** (H1').  $A, V \in L_{\text{loc}}^p(\mathbb{R}, \mathbb{R})$  with  $p \geq 2$  such that  $A = A_1 + A_2$ ,  $V = V_1 + V_2$ , where  $A_1, V_1$  have compact support and  $A_2, V_2 \in C^1(\mathbb{R}, \mathbb{R})$  fulfill

- i')  $A_2$  is supported away from 0 and  $\text{supp}(V_2) \subset \text{supp}(A_2)$ ,
- ii')  $\|V_2/A_2\|_\infty < 1$ ,
- iii') the derivative  $(V_2/A_2)'$  is bounded on  $\mathbb{R}$ .

Due to this assumptions we can choose  $\beta = V_2/A_2 = V_2/A$  and set  $\theta = \tanh^{-1} \beta$ ,  $\gamma^{-2} = 1 - \beta^2$  as above. Then, as in (29), we obtain

$$\begin{aligned} e^{-\sigma_2\theta/2} h e^{\sigma_2\theta/2} &= \gamma(1 - \sigma_2\beta)\sigma_1(-i\partial_1 - i\sigma_2\frac{\theta'}{2}) - (1 - \sigma_2\frac{V_2}{A})\sigma_2A + V_1 \\ &= M\left[\sigma_1(-i\partial_1) - \sigma_2A/\gamma + \gamma(1 + \sigma_2\beta)V_1 + \sigma_3\frac{\theta'}{2}\right] \end{aligned}$$

on  $C_0^\infty(\mathbb{R}, \mathbb{C}^2)$ , using again the notation  $M = \gamma(1 - \sigma_2\beta)$ . Beside of the residual terms  $\gamma(1 + \sigma_2\beta)V_1$ ,  $\sigma_3\theta'/2$ , the operator

$$\hat{h} := \sigma_1(-i\partial_1) - \sigma_2A/\gamma + \gamma(1 + \sigma_2\beta)V_1 + \sigma_3\frac{\theta'}{2} \quad \text{on } L^2((0, \infty), \mathbb{C}^2)$$

has a magnetic vector potential  $A/\gamma$ . As in the case  $A_2 < V_2$ , we conclude

**Lemma 2.** *Consider the operator  $h$  with potentials  $A, V$  satisfying (H1'). Then  $\mathcal{D}(\hat{h}) = \mathcal{D}(M\hat{h}) = e^{-\sigma_2\theta/2}\mathcal{D}(h)$  and the resolvent sets fulfill  $\varrho(h) = \varrho(M\hat{h})$ . For any  $z \in \varrho(h)$  holds*

$$\begin{aligned} (M\hat{h} - z)^{-1} &= e^{-\sigma_2\theta/2}(h - z)^{-1}e^{\sigma_2\theta/2} \quad \text{and} \\ \|(M\hat{h} - z)^{-1}\| &\leq \|(h - i)^{-1}\| \|e^{|\theta|}\|_\infty. \end{aligned}$$

**Remark 5.** *The same statement is valid for  $h_k$  with corresponding conditons and operators  $\hat{h}_k$ ,  $k \in \mathbb{Z} + \frac{1}{2}$ .*

## 5. HILBERT-SCHMIDT BOUNDS

In this section we prove Theorems 1 and 3. We have verified these results already for the Dirac operator with purely electric potentials in Proposition 1. To treat the general case we use the connection between  $h$  ( $h_k$ ) and  $\tilde{h}$  ( $\tilde{h}_k$ ) established in the previous section.

For the next Lemmas recall the definitions of the self-adjoint operators  $h$  and  $h_k$  and of  $\tilde{h}$  and  $\tilde{h}_k$  given in Sections 3 and 4, respectively.

**Lemma 3.** *Assume that  $A, V$  satisfy Hypothesis (H1). Then, there is a bounded operator  $S \equiv S(A, V)$  such that*

$$(\tilde{h} - i)^{-1} = U_2^*(-i\sigma_1\partial_x - i)^{-1}S,$$

where  $[U_2\psi](x) = \exp\left(i\sigma_1 \int_0^x (V_2/\gamma)(s)ds\right)\psi(x)$ .

*Proof.* Recall that

$$\tilde{h} = -i\sigma_1\partial_x - \gamma(1 + \sigma_2\beta)\sigma_2A_1 + V_1/\gamma + V_2/\gamma + \sigma_3\frac{\theta'}{2},$$

where  $\theta = \tanh^{-1}(A_2/V)$  has a uniformly bounded derivative. For shorthand notation we set  $W := -\gamma(1 + \sigma_2\beta)\sigma_2A_1 + V_1/\gamma$ . By the second resolvent identity we obtain

$$\begin{aligned} (\tilde{h} - i)^{-1} &= (-i\sigma_1\partial_x + W + V_2/\gamma - i)^{-1} \left[1 - \sigma_3\frac{\theta'}{2}(\tilde{h} - i)^{-1}\right] \\ &= U_2^*\left(-i\sigma_1\partial_x + \widetilde{W} - i\right)^{-1} U_2 \left[1 - \sigma_3\frac{\theta'}{2}(\tilde{h} - i)^{-1}\right], \end{aligned}$$

where  $\widetilde{W} = U_2 W U_2^*$ . By the assumptions  $|\widetilde{W}| \in L^p$  for some  $p \geq 2$ , which implies that  $\widetilde{W}$  is relatively compact with respect to  $-i\sigma_1\partial_x$  (use Kato-Seiler-Simon inequality). In particular,  $\widetilde{W}(-i\sigma_1\partial_x + \widetilde{W} - i)^{-1}$  is bounded. Finally, by the second

resolvent identity we have

$$\left(-i\sigma_1\partial_x + \widetilde{W} - i\right)^{-1} = \left(-i\sigma_1\partial_x - i\right)^{-1} \left[1 - \widetilde{W} \left(-i\sigma_1\partial_x + \widetilde{W} - i\right)^{-1}\right],$$

from which follows the claim.  $\square$

Throughout the rest of this section we write  $F := \sigma_2\theta/2 = \sigma_2A_2/(2V)$  (see Section 4).

*Proof of Theorem 1.* By a simple perturbational argument it suffices to prove the statement for  $\xi = 0$ , i.e. for  $h(0) = h$ . Using Equation (29) and Lemma 1 we compute

$$\begin{aligned} \mathbb{1}_I(h - i)^{-1} &= \mathbb{1}_I e^F (M\tilde{h} - i)^{-1} e^{-F} \\ &= \mathbb{1}_I e^F (\tilde{h} - i)^{-1} \left[ (\tilde{h} - i)(M\tilde{h} - i)^{-1} \right] e^{-F}, \end{aligned}$$

where the operator in [...] is bounded by Lemma 1 and the Closed Graph Theorem. Thus, we find some constant  $c$  such that

$$\|\mathbb{1}_I(h - i)^{-1}\|_{\text{HS}} \leq c \|\mathbb{1}_I(\tilde{h} - i)^{-1}\|_{\text{HS}}.$$

The claim is now a direct consequence of Lemmas 3 and Proposition 1.  $\square$

*Proof of Theorem 3.* Observe that

$$\mathbb{1}_I(h_k - i)^{-1} = e^F \mathbb{1}_I(\tilde{h}_k - i)^{-1} \left[ (\tilde{h}_k - i)(M\tilde{h}_k - i)^{-1} e^{-F} \right],$$

which implies, by the Closed Graph Theorem and Lemma 1, that

$$(32) \quad \|\mathbb{1}_I(h_k - i)^{-1}\|_{\text{HS}} \leq c \|\mathbb{1}_I(\tilde{h}_k - i)^{-1}\|_{\text{HS}},$$

for some constant  $c > 0$ . Here

$$\tilde{h}_k = \sigma_1(-i\partial_x) + \gamma(1 + \sigma_2\beta)\sigma_2\left(\frac{k}{x} - A_1\right) + V_1/\gamma + V_2/\gamma + \sigma_3\frac{\theta'}{2}.$$

In order to make the argument more transparent we write

$$\tilde{h}_k = \sigma_1(-i\partial_x) + \sigma_2\frac{k}{x} + W + V_2/\gamma,$$

where  $W = W_1 + W_2$  and

$$W_1 := V_1/\gamma - \gamma(1 + \sigma_2\beta)\sigma_2A_1,$$

which has compact support and is obviously in  $L^p(\mathbb{R}^+, \mathbb{C}^{2 \times 2})$  for some  $p > 2$ , and

$$W_2 := \gamma(1 + \sigma_2\beta)\sigma_2\frac{k}{x} - \sigma_2\frac{k}{x} + \sigma_3\frac{\theta'}{2} = \left((\gamma - 1) + \gamma\beta\sigma_2\right)\sigma_2\frac{k}{x} + \sigma_3\frac{\theta'}{2}.$$

Due to the support properties of  $A_2$  (recall that  $\beta = A_2/V$  and  $\gamma = (1 - \beta^2)^{-1/2}$ ), the function  $W_2$  is supported away from zero. In addition, observe that  $W_2$  is uniformly bounded and hence

$$(33) \quad \|\mathbb{1}_I(\tilde{h}_k - i)^{-1}\|_{\text{HS}} \leq c \|\mathbb{1}_I(\tilde{h}_k - W_2 - i)^{-1}\|_{\text{HS}}.$$

According to Corollary 3 (from Section 6)  $W_1$  is an infinitesimally small perturbation with respect to  $\tilde{h}_k - W = \sigma_1(-i\partial_x) + \sigma_2\frac{k}{x} + V_2/\gamma$  and therefore

$$(34) \quad \|\mathbb{1}_I(\tilde{h}_k - W_2 - i)^{-1}\|_{\text{HS}} \leq c \|\mathbb{1}_I(\tilde{h}_k - W - i)^{-1}\|_{\text{HS}}.$$

Let us define the self-adjoint operator

$$h_0 := \sigma_1(-i\partial_x) + V_2/\gamma$$

with domain,  $\mathcal{D}(h_0)$ , given by (19) for  $\alpha = 0$ . We can compare the resolvents of  $\tilde{h}_k - W$  and  $h_0$  as follows:

Define  $\chi \in C^\infty((0, \infty), [0, 1])$  such that  $\chi = 0$  on  $(0, \frac{1}{2})$  and  $\chi = 1$  on  $[1, \infty)$ . We compute

$$\begin{aligned} \mathbb{1}_I(\tilde{h}_k - W - i)^{-1} &= \mathbb{1}_I \chi(\tilde{h}_k - W - i)^{-1} \\ &= \mathbb{1}_I(h_0 - i)^{-1} \left[ (h_0 - i) \chi(\tilde{h}_k - W - i)^{-1} \right]. \end{aligned}$$

Observe that the operator in [...] is bounded by Remark 4 and the Closed Graph Theorem (and its norm will depend on  $|k|$ ). Thus there is a  $c_{|k|}$  such that

$$(35) \quad \left\| \mathbb{1}_I(\tilde{h}_k - W - i)^{-1} \right\|_{\text{HS}} \leq c_{|k|} \left\| \mathbb{1}_I(h_0 - i)^{-1} \right\|_{\text{HS}}.$$

This implies the result by Proposition 1.  $\square$

## 6. PROOF OF THEOREM 4

The main object of this section is the proof of Theorem 4 for the Dirac operators  $h_k$ . As we already mentioned  $h_k$  needs special care due to the  $k/x$ -singularity. Let us explain this a little further: Recall that for  $A = V = 0$

$$(36) \quad h_k = \sigma_1(-i\partial_x) + \sigma_2 \frac{k}{x} \quad \text{on } L^2((0, \infty), \mathbb{C}^2).$$

Observe that, for  $\varphi \in C_0^\infty((0, \infty), \mathbb{C}^2)$ ,

$$\begin{aligned} (37) \quad \|\varphi\|_{h_k}^2 &= \|\varphi'\|^2 + \|\varphi\|^2 - \left\langle \varphi, \sigma_3 \frac{k}{x^2} \varphi \right\rangle + \left\| \frac{k}{x} \varphi \right\|^2 \\ &\geq \|\varphi'\|^2 + \|\varphi\|^2 + (k^2 - |k|) \left\| \frac{1}{x} \varphi \right\|^2. \end{aligned}$$

Using this and the standard Hardy inequality on the half-line

$$(38) \quad \int_0^\infty |\varphi'(x)|^2 dx \geq \frac{1}{4} \int_0^\infty \frac{|\varphi'(x)|^2}{x^2} dx$$

one can show, for the case  $|k| > 1/2$ , that  $L^p$ -perturbations can be controlled by the  $h_k$ -graph norm, whenever  $p \geq 2$ . For the important case  $|k| = 1/2$ , this argument does not work since the Hardy inequality becomes critical. Instead, a version of the Hardy-Sobolev-Maz'ya inequality on the half-line, proven recently in [6], allows us to control  $L^p$ -perturbations, but only for  $p > 2$ . These observations, made precise in Theorem 5 and Corollary 3, enable us to show that the operator  $\mathbb{1}_{(0,R)}(h_k - i)^{-1}$  is compact, provided the potentials  $A, V$  are locally in  $L^p$  for  $p > 2$  (Corollary 4). The latter is an important ingredient for the proof of Theorem 4 at the end of this section.

**Remark 6.** In view of (37) and (38) it is clear that  $1/x$  is a perturbation with respect to  $(-i\sigma_1\partial_x + \sigma_2 \frac{k}{x})$ , provided the strict inequality  $|k| > 1/2$  holds.

**Theorem 5.** For  $|k| \geq \frac{1}{2}$  consider  $h_k$  with  $A = V = 0$ . Then, any multiplication operator  $M \in L^p((0, \infty), \mathbb{C}^{2 \times 2})$  with  $p > 2$  is infinitesimally  $h_k$ -bounded. In addition, any multiplication operator  $M \in L^2((0, \infty), \mathbb{C}^{2 \times 2})$  is infinitesimally  $h_k$ -bounded for  $|k| > \frac{1}{2}$ .

*Proof.* Let  $\varphi \in C_0^\infty((0, \infty), \mathbb{C}^2)$  and  $|k| \geq \frac{1}{2}$ , then

$$\begin{aligned}
 \|\varphi\|_{h_k}^2 &= \|\varphi'\|^2 + \|\varphi\|^2 - \left\langle \varphi, \sigma_3 \frac{k}{x^2} \varphi \right\rangle + \left\| \frac{k}{x} \varphi \right\|^2 \\
 (39) \quad &\geq \|\varphi'\|^2 + \|\varphi\|^2 + (k^2 - |k|) \left\| \frac{1}{x} \varphi \right\|^2 \\
 &\geq \|\varphi'\|^2 + \|\varphi\|^2 - e^{-4(|k| - \frac{1}{2})^2} \frac{1}{4} \left\| \frac{1}{x} \varphi \right\|^2.
 \end{aligned}$$

Using the one-dimensional Sobolev inequality  $\|\varphi\|_\infty^2 \leq \kappa \|\varphi\|^2 + \kappa^{-1} \|\varphi'\|^2$  valid for  $\kappa > 1$ , we get that

$$(40) \quad \|\varphi\|_{h_k}^2 \geq (1 - \mu(k)) \kappa \|\varphi\|_\infty^2 + (1 - \kappa^2) \|\varphi\|^2 + \mu(k) \left( \|\varphi'\|^2 - \frac{1}{4} \left\| \frac{1}{x} \varphi \right\|^2 \right),$$

where  $\mu(k) := e^{-4(|k| - \frac{1}{2})^2} \in (0, 1]$ . (Note that the first term on the right hand side of (40) equals zero when  $k = 1/2$ .) By the Hardy-Sobolev-Maz'ya inequality on the half-line (see [6, Thm. 1.2]) we obtain, for  $q \in (2, \infty)$  and  $\theta = \frac{1}{2}(1 - 2q^{-1})$ , a constant  $c_\theta$  (depending only on  $\theta$ ) such that

$$\begin{aligned}
 c_\theta \|\varphi\|_q^2 &\leq \left( \|\varphi'\|^2 - \frac{1}{4} \left\| \frac{1}{x} \varphi \right\|^2 \right)^\theta (\|\varphi\|^2)^{1-\theta} \\
 &\leq \epsilon \theta \left( \|\varphi'\|^2 - \frac{1}{4} \left\| \frac{1}{x} \varphi \right\|^2 \right) + (1 - \theta) \epsilon^{-\frac{\theta}{1-\theta}} \|\varphi\|^2.
 \end{aligned}$$

In the last step we use Young's inequality with  $\epsilon \in (0, 1)$ . Combining this with (40) we conclude that

$$(41) \quad \mu(k) c_\theta \|\varphi\|_q^2 + (1 - \mu(k)) \kappa \epsilon \theta \|\varphi\|_\infty^2 \leq \epsilon \theta \|\varphi\|_{h_k}^2 + c(\epsilon, \kappa, \theta) \|\varphi\|^2.$$

For  $M \in L^p((0, \infty), \mathbb{C}^{2 \times 2})$  with  $p > 2$  we choose  $\theta = p^{-1}$  (hence  $p^{-1} + q^{-1} = 1/2$ ), then (41) yields

$$(42) \quad \|M\varphi\|^2 \leq \|M\|_p^2 \|\varphi\|_q^2 \leq \|M\|_p^2 (\mu(k) c_\theta)^{-1} (\epsilon \theta \|\varphi\|_{h_k}^2 + c(\epsilon, \kappa, \theta) \|\varphi\|^2)$$

for any  $\epsilon \in (0, 1]$ . If  $M$  is a  $L^2$ -function and  $|k| > \frac{1}{2}$  we use again (41) (dropping the first term) to obtain

$$(43) \quad \|M\varphi\|^2 \leq \|M\|_2^2 \|\varphi\|_\infty^2 \leq \frac{\kappa^{-1}}{(1 - \mu(k))} \|M\|_2^2 \|\varphi\|_{h_k}^2 + \tilde{c}(\epsilon, \kappa, \theta) \|\varphi\|^2.$$

Since  $h_k$  is essentially self-adjoint on  $C_0^\infty((0, \infty), \mathbb{C}^2)$  inequalities (42) and (43) imply the claim.  $\square$

**Corollary 3.** For  $|k| \geq \frac{1}{2}$  consider  $h_k$  with  $A, V \in L_{\text{loc}}^p((0, \infty), \mathbb{R})$  for some  $p > 2$ . Then any multiplication operator  $M \in L^s((0, \infty), \mathbb{C}^{2 \times 2})$ ,  $s > 2$ , with compact support is infinitesimally  $h_k$ -bounded.

*Proof.* Let  $\chi \in C^\infty(\mathbb{R}^+, [0, 1])$  be a smooth cut-off function which equals 1 on the support of  $M$  and vanishes for large  $x$ . Then, for any  $\varphi \in C_0^\infty(\mathbb{R}^+, \mathbb{C}^2)$  and  $\epsilon \in (0, 1)$  we find, by Theorem 5, a constant  $c_\epsilon$  such that

$$(44) \quad \|M\varphi\| = \|M\chi\varphi\| \leq \epsilon \left\| \left( -i\sigma_1 \partial_x + \sigma_2 \frac{k}{x} \right) \chi \varphi \right\| + c_\epsilon \|\varphi\|.$$

Let us write  $W := V - \sigma_2 A \in L^p_{\text{loc}}((0, \infty), \mathbb{R})$ . Using again Theorem 5 we find a constant  $c > 0$  with

$$\begin{aligned} \|(-i\sigma_1\partial_x + \sigma_2\frac{k}{x})\chi\varphi\| &\leq \|h_k\chi\varphi\| + \|W\chi\varphi\| \\ &\leq \|h_k\chi\varphi\| + \frac{1}{2}\|(-i\sigma_1\partial_x + \sigma_2\frac{k}{x})\chi\varphi\| + c\|\varphi\| \\ &\leq \|h_k\varphi\| + \frac{1}{2}\|(-i\sigma_1\partial_x + \sigma_2\frac{k}{x})\chi\varphi\| + (c + \|\chi'\|_\infty)\|\varphi\|. \end{aligned}$$

We get the desired result by combining this with (44).  $\square$

**Corollary 4.** *For  $|k| \geq \frac{1}{2}$  consider  $h_k$  with  $A, V \in L^p_{\text{loc}}((0, \infty), \mathbb{R})$  for some  $p > 2$ . Then,  $h_k$  is a locally compact operator, i.e. for any  $R > 0$  the operator  $\mathbb{1}_{(0, R)}(h_k - i)^{-1}$  is compact.*

*Proof.* For  $R > 0$  let  $\chi \in C^\infty([0, \infty), [0, 1])$  be a smooth cutoff-function with  $\chi(x) = 1$  for  $x \leq R$  and  $\chi(x) = 0$  for  $x \geq R+1$ . We compare  $h_k$  with the reference operator

$$h_{\text{ref}} = \sigma_1(-i\partial_x) + \sigma_2\frac{1}{x} \quad \text{on} \quad L^2((0, \infty), \mathbb{C}^2),$$

which is known to be locally compact (see e.g. [14]). We compute the resolvent difference

$$\begin{aligned} \chi^2 \frac{1}{h_{\text{ref}} - i} - \frac{1}{h_k - i} \chi^2 &= \frac{1}{h_k - i} ((h_k - i)\chi^2 - \chi^2(h_{\text{ref}} - i)) \frac{1}{h_{\text{ref}} - i} \\ &= \frac{1}{h_k - i} ((V - \sigma_2 A)\chi^2 - 2i\sigma_1\chi\chi') \frac{1}{h_{\text{ref}} - i} \\ &\quad + (k-1) \frac{1}{h_k - i} x^{-1/4} \chi^2 \sigma_2 x^{-3/4} \frac{1}{h_{\text{ref}} - i}. \end{aligned}$$

Using that  $h_{\text{ref}}$  is locally compact and that  $(V - \sigma_2 A)\chi$ ,  $x^{-1/4}\chi$ , and  $2i\sigma_1\chi'$  are relatively  $h_k$ -bounded (see Corollary 3), it suffices to show that

$$\chi x^{-3/4} \frac{1}{h_{\text{ref}} - i}$$

is a compact operator. To this end we first recall that  $x^{-2}$  is bounded with respect to  $h_{\text{ref}}^2$  in the sense of quadratic forms (see Remark 6). Since exponentiating to the power  $3/4$  is operator monotonic we conclude that  $x^{-3/4}|h_{\text{ref}} - i|^{-3/4}$  is bounded. Therefore, by the relation

$$\begin{aligned} \chi x^{-3/4} \frac{1}{h_{\text{ref}} - i} &= x^{-3/4} \left| \frac{1}{h_{\text{ref}} - i} \right|^{3/4} \text{sgn} \left( \frac{1}{h_{\text{ref}} - i} \right) \left| \frac{1}{h_{\text{ref}} - i} \right|^{1/4} \chi + \\ &\quad x^{-3/4} \frac{1}{h_{\text{ref}} - i} (-i\sigma_1\chi') \frac{1}{h_{\text{ref}} - i} \end{aligned}$$

it suffices to show that  $\chi|h_{\text{ref}} - i|^{-1/4}$  is compact. This, however, follows from the identity

$$\begin{aligned} \left| \frac{1}{h_{\text{ref}} - i} \right|^{1/4} \chi &= \left( \frac{1}{h_{\text{ref}}^2 + 1} \right)^{1/8} \chi \\ &= B(\frac{7}{8}, \frac{1}{8})^{-1} \int_0^\infty \frac{1}{h_{\text{ref}}^2 + s + 1} \chi \frac{1}{s^{1/8}} ds \end{aligned}$$

(here  $B(x, y)$  denotes the beta function), since a Riemannian integral of compact operators is also compact.  $\square$

**Remark 7.** *If we assume further that  $|k| > 1/2$  then, in view of Theorem 5, the statements of Corollaries 3 and 4 are also valid for  $L^2$ -perturbations.*

*Proof of Theorem 4.* In this proof we slightly modify the argument of [10, Theorem 6.2] for the operator defined on the half-line. Let  $\mathcal{B}(\mathbb{R})$  denote the Borel  $\sigma$ -algebra on  $\mathbb{R}$ . Given any  $\psi \in L^2(\mathbb{R}^+, \mathbb{C}^2)$  we write its associated spectral measure (with respect to  $h_k$ ) as

$$\mu_\psi : \mathcal{B}(\mathbb{R}) \rightarrow [0, \infty), \quad \Omega \mapsto \langle \psi, \mathbb{1}_\Omega(h_k)\psi \rangle.$$

Since  $\mu_\psi$  is absolutely continuous with respect to the Lebesgue measure it can be decomposed as a sum of mutually singular measures  $\mu_\psi = \mu_{\psi,1} + \mu_{\psi,2}$ , where  $\mu_{\psi,2}(\mathbb{R}) < \|\psi\|^2/4$  and  $\mu_{\psi,1}$  is a uniformly Lipschitz continuous measure, i.e. there is a constant  $C > 0$  such that for any interval with Lebesgue measure  $|I| < 1$ ,  $\mu_{\psi,1}(I) < C|I|$  (this can be verified decomposing the Radon-Nykodym derivative,  $f_\psi$ , associated to  $\mu_\psi$  as  $f_\psi = f_\psi \mathbb{1}_{\{f_\psi < \alpha\}} + f_\psi \mathbb{1}_{\{f_\psi > \alpha\}}$  for  $\alpha > 0$  sufficiently large; for a more general statement involving uniform  $\alpha$ -Hölder continuity see [10, Theorem 4.2]). For  $j \in \{1, 2\}$  define  $\psi_j := \mathbb{1}_{S_j}(h_k)\psi$  where  $S_j \subset \mathbb{R}$  is the support of the measure  $\mu_{\psi,j}$ . Then, for any  $\Omega \in \mathcal{B}(\mathbb{R})$ , we have

$$\begin{aligned} \mu_{\psi_j}(\Omega) &= \langle \psi_j, \mathbb{1}_\Omega(h_k)\psi_j \rangle = \langle \psi, \mathbb{1}_{\Omega \cap S_j}(h_k)\psi \rangle \\ &= \mu_\psi(\Omega \cap S_j) = \mu_{\psi,1}(\Omega \cap S_j) + \mu_{\psi,2}(\Omega \cap S_j) = \mu_{\psi,j}(\Omega), \end{aligned}$$

where in the last equality we use that the measures  $\mu_{\psi,j}$  are disjointly supported. Thus, we get that  $\mu_\psi = \mu_{\psi_1} + \mu_{\psi_2}$ . Note that  $\psi_1 \neq 0$  since

$$\|\psi_1\|^2 = \mu_{\psi_1}(\mathbb{R}) = \|\psi\|^2 - \mu_{\psi_2}(\mathbb{R}) \geq 3\|\psi\|^2/4.$$

For any  $R > 1$  we have

$$(45) \quad \|x^{p/2}e^{-i h_k t}\psi\|^2 \geq \|R^{p/2}\mathbb{1}_{(R,\infty)}e^{-i h_k t}\psi\|^2 \geq R^p(\|\psi\|^2 - \|\mathbb{1}_{(0,R)}e^{-i h_k t}\psi\|^2).$$

We observe that since  $\psi_1$  is orthogonal to  $\psi_2$  the triangular inequality yields

$$(46) \quad \begin{aligned} \|\mathbb{1}_{(0,R)}e^{-i h_k t}\psi\|^2 &\leq 2\|\mathbb{1}_{(0,R)}e^{-i h_k t}\psi_1\|^2 + 2\|\psi_2\|^2 \\ &\leq 2\|\mathbb{1}_{(0,R)}e^{-i h_k t}\psi_1\|^2 + \frac{1}{2}\|\psi\|^2. \end{aligned}$$

In order to use Theorem 3 we replace the cut-off function above by one supported away from zero. Note that by Corollary 4 and the RAGE theorem we find a  $T_0 > 0$  such that for all  $T > T_0$  one has that

$$\begin{aligned} \langle \|\mathbb{1}_{(0,R)}e^{-i h_k t}\psi_1\|^2 \rangle_T &= \langle \|\mathbb{1}_{(0,1)}e^{-i h_k t}\psi_1\|^2 \rangle_T + \langle \|\mathbb{1}_{(1,R)}e^{-i h_k t}\psi_1\|^2 \rangle_T \\ &\leq \frac{1}{8}\|\psi\|^2 + \langle \|\mathbb{1}_{(1,R)}e^{-i h_k t}\psi_1\|^2 \rangle_T. \end{aligned}$$

Combining the latter bound with (46) and (45) we readily obtain, for  $T > T_0$ ,

$$(47) \quad \langle \|x^{p/2}e^{-i h_k t}\psi\|^2 \rangle_T \geq R^p\left(\frac{1}{4}\|\psi\|^2 - 2\langle \|\mathbb{1}_{(1,R)}e^{-i h_k t}\psi_1\|^2 \rangle_T\right).$$

Next we recall (see [10, Theorem 3.2]) that given a self-adjoint operator  $H$  and a Hilbert-Schmidt operator  $A$  one finds a constant  $c_\varphi$  such that

$$(48) \quad \langle \|Ae^{-itH}\varphi\| \rangle_T \leq c_\varphi \|A\|_{\text{HS}}^2 T^{-1}$$

provided the  $H$ -spectral measure associated to  $\varphi$  is Lipschitz continuous. Applying this and Theorem 3 we obtain

$$\begin{aligned}
 \langle \|\mathbb{1}_{(1,R)} e^{-i h_k t} \psi_1\|^2 \rangle_T &= \langle \|\mathbb{1}_{(1,R)} \mathbb{1}_\Delta(h_k) e^{-i h_k t} \psi_1\|^2 \rangle_T \\
 (49) \qquad \qquad \qquad &\leq c_{\psi_1} T^{-1} \|\mathbb{1}_{(1,R)} (h_k - i)^{-1} (h_k - i) \mathbb{1}_\Delta(h_k)\|_{\text{HS}}^2 \\
 &\leq c_{\psi_1} c_\Delta C_k R T^{-1} \equiv \frac{1}{2} \widehat{C}_k(\psi_1, \Delta, k) R T^{-1},
 \end{aligned}$$

where  $c_\Delta = \|(h_k - i) \mathbb{1}_\Delta\|^2$ . Hence, using the latter bound in (47) we get, for  $T > T_0$ , that

$$\begin{aligned}
 \langle \|x^{p/2} e^{-i h_k t} \psi\|^2 \rangle_T &\geq R^p \left( \frac{1}{4} \|\psi\|^2 - \widehat{C}_k(\psi_1, \Delta, k) R T^{-1} \right) \\
 &= \frac{1}{8^{p+1} \widehat{C}_k(\psi_1, \Delta, k)^p} \|\psi\|^{2p+2} T^p,
 \end{aligned}$$

where in the last equality we have chosen

$$R \equiv R(T) = \frac{\|\psi\|^2}{8 \widehat{C}_k(\psi_1, \Delta, k)} T.$$

Finally note that the inequality (12) is trivially fulfilled for finite  $T \in [0, T_0]$  by just choosing the constant  $C_k(\psi_1, \Delta, k)$  suitably.  $\square$

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#### APPENDIX A. REMARKS ON SELF-ADJOINTNESS OF DIRAC OPERATORS

In this section we state and prove some facts concerning self-adjointness of the one-dimensional Dirac operators discussed in Section 3. These facts are well known, however, they are not easy to find in the standard literature.

**Proposition 2.** *Let  $|k| \geq \frac{1}{2}$ , then  $h_k$  is in the limit point case at 0.*

*Proof.* It suffices to show that there is a solution to the eigenvalue problem

$$(50) \qquad h_k \varphi = \lambda \varphi$$

which is not square-integrable at 0. According to [3, Theorem 1] (see also [21]), for  $k \geq \frac{1}{2}$ , there is a unique solution to (50) with the asymptotic behaviour

$$u(r) = (o(1), 1 + o(1))^T x^k \quad \text{as } x \rightarrow 0.$$

(Note that [3, Theorem 1] is only stated for the case  $A = 0$ , however, the same argument applies provided  $A$  is integrable at zero.) Let  $w$  be a linear independent solution of (50) such that the Wronski determinant  $W(u, w) := u_1 w_2 - u_2 w_1 \equiv 1$ . Assume that  $\liminf_{x \rightarrow 0} |w(x)| x^k = 0$ , then clearly  $\liminf_{x \rightarrow 0} W(u, w)(x) = 0$  which is a contradiction. Hence  $w$  can not be square-integrable at 0. An analogous argument holds also when  $k \leq -1/2$ .  $\square$



**Lemma 4.** *Let  $k \in \mathbb{Z} + \frac{1}{2}$ , then  $C_0^\infty((0, \infty), \mathbb{C}^2)$  is dense in  $\mathcal{D}_0(h_k)$  with respect to the  $h_k$ -graph norm. The analogous statement holds for  $C_0^\infty(\mathbb{R}, \mathbb{C}^2) \subset \mathcal{D}_0(h)$  in the  $h$ -graph norm.*

*Proof.* We give the details of the proof only for the operator defined on the whole real line. First note that clearly  $C_0^\infty(\mathbb{R}, \mathbb{C}^2)$  is a subset of  $\mathcal{D}_0(h)$ . Let  $\psi \in \mathcal{D}_0(h)$  and let  $K$  be a compact set which contains the support of  $\psi$ . Since  $\psi \in C_0(\mathbb{R}, \mathbb{C}^2)$  and  $(V - \sigma_2 A) \in L_{\text{loc}}^2$  we have that

$$-i\sigma_1\psi' = h\psi - (V - \sigma_2 A)\psi \in L^2(\mathbb{R}, \mathbb{C}^2),$$

which implies that  $\psi \in H^1(\mathbb{R}, \mathbb{C}^2)$ . Let  $(\psi_n)_{n \in \mathbb{N}} \subset C_0^\infty$  be a sequence of mollifiers of  $\psi$  whose support is also contained in  $K$ . We estimate

$$\begin{aligned} \|\psi - \psi_n\|_h^2 &= \|\psi - \psi_n\|_2^2 + \|h(\psi - \psi_n)\|_2^2 \leq \|\psi - \psi_n\|_{H^1}^2 + \|(V - \sigma_2 A)(\psi - \psi_n)\|_2^2 \\ &\leq \|\psi - \psi_n\|_{H^1}^2 + \|\psi - \psi_n\|_\infty^2 \|(V - \sigma_2 A)\mathbb{1}_K\|_2^2. \end{aligned}$$

By the  $H^1$ -convergence of mollifiers we know that  $\|\psi - \psi_n\|_{H^1}^2 \rightarrow 0$ . Moreover, the Sobolev inequality in dimension one implies that  $\|\psi - \psi_n\|_\infty^2 \rightarrow 0$ . Hence,  $\psi_n$  converges to  $\psi$  in the graph norm, as claimed. The argument for the operator  $h_k$  is completely analogous. Just note that  $k/x \in L_{\text{loc}}^2((0, \infty))$ .  $\square$

## APPENDIX B. COMPUTATION OF A RESOLVENT KERNEL

In order to compute a resolvent kernel of the operator  $\sigma_1(-i\partial_x)$  on  $L^2((0, \infty), \mathbb{C}^2)$  with the boundary condition  $\psi_1(0) = 0$ , we use the unitary matrix  $\hat{U} = \frac{1}{\sqrt{2}}(\mathbb{1} + i\sigma_3)$  to transform the problem to the operator

$$\sigma_2(i\partial_x) = \frac{1}{2}(\mathbb{1} + i\sigma_3)\sigma_1(-i\partial_x)(\mathbb{1} - i\sigma_3) = \hat{U}\sigma_1(-i\partial_x)\hat{U}^*$$

on  $L^2((0, \infty), \mathbb{C}^2)$  with the same boundary conditions. The corresponding relation for the resolvent is

$$(51) \quad \frac{1}{\sigma_1(-i\partial_x) - i} = \hat{U}^* \frac{1}{\sigma_2(i\partial_x) - i} \hat{U}.$$

We note that, by [23, Section 15.5], the kernel of the resolvent (51) can be given in terms of a fundamental system  $u_1(z, \cdot), u_2(z, \cdot)$  of the ODE  $(\sigma_1(-i\partial_x) - z)u = 0$ . For  $z \in \mathbb{C} \setminus \mathbb{R}$  this is given as follows

$$\frac{1}{\sigma_2(i\partial_x) - z}(x_1, x_2) = \begin{cases} \sum_{j,k=1}^2 m_{j,k}^+(z) \overline{u_j(\bar{z}, x_1)} u_k^T(z, x_2) & \text{if } x_1 > x_2 > 0, \\ \sum_{j,k=1}^2 m_{j,k}^-(z) \overline{u_j(\bar{z}, x_1)} u_k^T(z, x_2) & \text{if } x_2 > x_1 > 0, \end{cases}$$

where  $m^+(z), m^-(z)$  are  $2 \times 2$  matrices whose coefficients are given in terms of certain complex numbers  $m_a(z), m_b(z)$  that are chosen such that  $m_a(z)u_1(z, \cdot) + u_2(z, \cdot)$  fulfill the boundary condition at 0 and  $m_b(z)u_1(z, \cdot) + u_2(z, \cdot)$  is square-integrable at  $\infty$  (see [23, Section 15.5]). Hence, using the fundamental system

$$u_1(z, x) = \begin{pmatrix} \cos zx \\ \sin zx \end{pmatrix}, \quad u_2(z, x) = \begin{pmatrix} -\sin zx \\ \cos zx \end{pmatrix},$$

we obtain that  $m_a(i) = 0, m_b(i) = i$ . Therefore,

$$m^+(i) = \begin{pmatrix} 0 & -1 \\ 0 & i \end{pmatrix}, \quad m^-(i) = \begin{pmatrix} 0 & 0 \\ -1 & i \end{pmatrix}.$$

In addition, using that

$$u_1(i, x) = \overline{u_1(-i, x)} = \begin{pmatrix} \cosh x \\ i \sinh x \end{pmatrix}, \quad u_2(i, x) = \overline{u_2(-i, x)} = \begin{pmatrix} -i \sinh x \\ \cosh x \end{pmatrix},$$

we get the explicit resolvent kernel

$$\frac{1}{\sigma_2(i \partial_x) - i}(x_1, x_2) = \begin{cases} e^{-x_1} \begin{pmatrix} i \sinh x_2 & -\cosh x_2 \\ \sinh x_2 & i \cosh x_2 \end{pmatrix} & \text{if } x_1 > x_2 > 0, \\ e^{-x_2} \begin{pmatrix} i \sinh x_1 & \sinh x_1 \\ -\cosh x_1 & i \cosh x_1 \end{pmatrix} & \text{if } x_2 > x_1 > 0. \end{cases}$$

Finally, by (51) we get (24).

#### APPENDIX C. PROOF OF COROLLARIES 1 AND 2

We observe that in the case of a translation symmetry in  $x_2$ -direction, we can write

$$\langle \| |x_1|^{p/2} e^{-i t H} \psi \|^2 \rangle_T = \int_{-\infty}^{\infty} \langle \| |x_1|^{p/2} e^{-i t h(\xi)} \widehat{\psi}(\cdot, \xi) \|^2 \rangle_T d\xi$$

By assumption (13) we find  $\xi_0 > 0$  large enough such that

$$M := \{ \xi \in [-\xi_0, \xi_0] \mid \widehat{\psi}(\cdot, \xi) \neq 0, \widehat{\psi}(\cdot, \xi) \in P_{ac}(h(\xi)) L^2(\mathbb{R}, \mathbb{C}^2) \}$$

has non-zero Lesbegue measure. Using that

$$\mathbb{1}_\Delta(H) = \int_{\mathbb{R}}^{\oplus} \mathbb{1}_\Delta(h(\xi)) d\xi$$

(see [13, Theorem XIII.85]), we conclude  $\widehat{\psi}(\cdot, \xi) \in \mathbb{1}_\Delta(h(\xi)) L^2(\mathbb{R}, \mathbb{C}^2)$  for a.e.  $\xi \in \mathbb{R}$  if  $\psi \in \mathbb{1}_\Delta(H) L^2(\mathbb{R}^2, \mathbb{C}^2)$ . Hence, Theorem 2 implies

$$\langle \| |x_1|^{p/2} e^{-i t H} \psi \|^2 \rangle_T \geq \int_M \langle \| |x_1|^{p/2} e^{-i t h(\xi)} \widehat{\psi}(\cdot, \xi) \|^2 \rangle_T d\xi \geq T^p \int_M C_\xi(\psi, \Delta, p) d\xi,$$

which give us the desired bound since  $\int_M C_\xi(\psi, \Delta, p) d\xi =: C(\psi, \Delta, p) > 0$ .

Now let us consider the case, when  $H$  is spherically symmetric and fulfills the assumptions of Corollary 1. We write

$$H \cong \bigoplus_{k \in \mathbb{Z} + \frac{1}{2}} h_k,$$

and use for

$$(52) \quad \psi = \sum_{j \in \mathbb{Z}} \psi_j \in \bigoplus_{j \in \mathbb{Z}} P_{ac}(h_j) L^2(\mathbb{R}^+, \mathbb{C}^2)$$

non-zero the estimate

$$\| |\mathbf{x}|^p e^{-i t H} \psi \|^2 = \sum_{j \in \mathbb{Z}} \| r^p e^{-i t h_j} \psi_j \|^2 \geq \sum_{j=-l}^l \| r^p e^{-i t h_j} \psi_j \|^2,$$

where  $l \in \mathbb{N}$  is chosen to be so large that  $\sum_{j=-l}^l \|\psi_j\|^2 \geq \frac{1}{2} \|\psi\|$ . Observing that  $\mathbb{1}_\Delta(H) = \bigoplus_{j \in \mathbb{Z}} \mathbb{1}_\Delta(h_j)$ , we deduce from Theorem 4 and (52) that

$$\| |\mathbf{x}|^p e^{-itH} \psi \|^2 \geq \sum_{j=-l}^l C_j(\psi_j, \Delta, p) T^p = C(\psi, \Delta, p) T^p$$

(where we set  $C_j(\psi_j, \Delta, p) = 0$  if  $\psi_j = 0$ ), with  $C(\psi, \Delta, p) > 0$ .

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